EDN: NEOBTF
УДК 517

On Periodic Bilinear Threshold GARCH models

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Received 23.09.2023, received in revised form 31.10.2023, accepted 14.02.2024

Abstract. Periodic Generalized Autoregressive Conditionally Heteroscedastic ($PGARCH$) models were introduced by Bollerslev et Ghysels. These models have gained considerable interest and continued to attract the attention of researchers. This paper is devoted to extensions of the standard bilinear threshold $GARCH$ ($BLTGARCH$) model to periodically time-varying coefficients ($PBLTGARCH$) one. In this class of models, the parameters are allowed to switch between different regimes. Moreover, these models are allowed to integrate asymmetric effects in the volatility. Firstly, we give necessary and sufficient conditions ensuring the existence of stationary solutions (in periodic sense). Secondly, a quasi maximum likelihood (QML) estimation approach for estimating $PBLTGARCH$ model is developed. More precisely, the strong consistency and the asymptotic normality of the estimator are studied given mild regularity conditions, requiring strict stationarity and the finiteness of moments of some order for the errors term. The finite-sample properties of $QMLE$ are illustrated by a Monte Carlo study. Finally our proposed model is applied to model the exchange rates of the Algerian Dinar against the single European currency ($Euro$).

Keywords: periodic bilinear threshold $GARCH$ models, Strictly periodically stationary, Gaussian QML estimator.

Citation: W. Slimani, I. Lescheb, M. Cherfaoui, On Periodic Bilinear Threshold $GARCH$ models, J. Sib. Fed. Univ. Math. Phys., 2024, 17(3), 334–346. EDN: NEOBTF.

1. Introduction and preliminaries

In recent years, many papers discussed the periodic generalized autoregressive conditionally heteroskedastic models ($PGARCH_s$) process introduced by Bollerslev and Ghysels [10]. This process has been proved to be a power tool for modeling and forecasting many non stationary time series, which makes a distinctive by a stochastic conditional variance with periodic dynamics. Generally, by $PGARCH_s$ process we mean a discrete-time strictly stationary process

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$(\varepsilon_t, t \in \mathcal{Z}), \mathcal{Z} = \{0, \pm 1, \pm 2, \dots,\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ and satisfying the factorization

$$\varepsilon_t = h_t e_t, \tag{1.1}$$

Here, the innovation process $(e_t, t \in \mathcal{Z})$ is independent and identically distributed sequence with zero mean and unit variance $(i.i.d\,(0,1))$ defined on the same probability space $(\Omega, \mathcal{F}, P)$ and time-varying coefficients "volatility" process $(h_t, t \in \mathcal{Z})$ satisfy the recursion

$$h_t^2 = \alpha_0\left(t\right) + \sum_{i=1}^{q} \alpha_i\left(t\right) \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j\left(t\right) h_{t-j}^2, \tag{1.2}$$

where $(\alpha_i\left(t\right), 0 \leqslant i \leqslant q)$ and $(\beta_j\left(t\right), 0 \leqslant j \leqslant p)$ are non negative periodic functions with period s with $\alpha_0\left(t\right) > 0$. $PGARCH_s$ model is potentially more efficient than the standard one. It becomes increasingly important and an efficient tool to model seasonal asset returns of stocks, exchange rates and other financial time series and continues to gain a growing interest of researchers (see Ghezal [1] and Lescheb [4]). This interest is due to its multiple advantages; for instance, among others, it is able to capture the stylized facts, e.g., volatility clustering, leptokurticity, dependency without correlation and tail heaviness. However, in some asymmetric financial datasets exhibiting the so-called leverage effect characterized by $Cov\left(e_{t-k}, h_t^2\right) < 0$, for some $k > 0$, the $PGARCHs$ models are unable to model such data without further extensions. This finding led Rodriguez and Ruiz [6] to study five of the most popular specifications of the time-invariant asymmetric volatility process $(h_t, t \in \mathbb{Z})$ with leverage effect, namely, the generalized quadratic $ARCH(GQARCH)$, the threshold $GARCH$ $(TGARCH)$, the $GJR-GARCH(GJR)$, the exponential $GARCH(EGARCH)$ and the asymmetric power $GARCH(APGARCH)$ models. These models are important in modelling, forecasting and capturing the asymmetry of the volatility and hence are purported to be able to capture the leverage. Beside the above mentioned models, Choi et al [7] have recently introduced the so-called bilinear threshold $GARCH$ $(BLTGARCH)$ model defined by Equation (1.1) with time-invariant coefficients volatility process, i.e.,

$$h_t^2 = \alpha_0 + \sum_{i=1}^{q} \left(\alpha_i \varepsilon_{t-i}^{+2} + \beta_i \varepsilon_{t-i}^{-2}\right) + \sum_{k=1}^{d} \left(b_k \varepsilon_{t-k}^{+} + \omega_k \varepsilon_{t-k}^{-}\right) h_{t-k} + \sum_{j=1}^{p} \gamma_j h_{t-j}^2, \tag{1.3}$$

where $\varepsilon_n^+ = \max\left(\varepsilon_n, 0\right), \varepsilon_n^- = \min\left(\varepsilon_n, 0\right), \varepsilon_n^{+2} = \left(\varepsilon_n^+\right)^2, \varepsilon_n^{-2} = \left(\varepsilon_n^-\right)^2$ and $d = p \wedge q$. This paper is fundamentally interested with non-stationary $BLTGARCH$ models in which the parameters are periodic in t with period s. As a result, we will provide a periodic $BLTGARCH(q, d, p)$ model $(PBLTGARCH_s)$ defined by (1.1) and

$$\begin{aligned} h_t^2 = \ & \alpha_0\left(t\right) + \sum_{i=1}^{q} \left(\alpha_i\left(t\right) \varepsilon_{t-i}^{+2} + \beta_i\left(t\right) \varepsilon_{t-i}^{-2}\right) + \sum_{k=1}^{d} \left(b_k\left(t\right) \varepsilon_{t-k}^{+} + \omega_k\left(t\right) \varepsilon_{t-k}^{-}\right) h_{t-k} \\ & + \sum_{j=1}^{p} \gamma_j\left(t\right) h_{t-j}^2. \end{aligned} \tag{1.4}$$

In (1.4), the functions $(\alpha_i\left(t\right), 0 \leqslant i \leqslant q)$, $(\beta_i\left(t\right), 1 \leqslant i \leqslant q)$, $(b_k\left(t\right), 1 \leqslant k \leqslant d)$, $(\omega_k\left(t\right), 1 \leqslant k \leqslant d)$ and $(\gamma_j\left(t\right), 1 \leqslant j \leqslant p)$ are periodic with period $s \geqslant 1$. Moreover, $(\alpha_i\left(t\right), 0 \leqslant i \leqslant q)$, $(\beta_i\left(t\right), 1 \leqslant i \leqslant q)$, $(\gamma_j\left(t\right), 1 \leqslant j \leqslant p)$ are non negative sequences with $\alpha_0(.) > 0$, whereas the functions $(b_k\left(t\right), 1 \leqslant k \leqslant d)$, $(\omega_k\left(t\right), 1 \leqslant k \leqslant d)$ have values in $(-\infty, +\infty)$. So, by transforming t into $st + v$ and setting $\varepsilon_t\left(v\right) = \varepsilon_{st+v}$, $h_t\left(v\right) = h_{st+v}$ and $e_t\left(v\right) = e_{st+v}$, then (1.4) may be equivalently written in periodic version as

$$\begin{aligned} h_t^2\left(v\right) & = \alpha_0\left(v\right) + \sum_{i=1}^{q} \left(\alpha_i\left(v\right) \varepsilon_t^{+2}\left(v-i\right) + \beta_i\left(v\right) \varepsilon_t^{-2}\left(v-i\right)\right) \\ & + \sum_{k=1}^{d} \left(b_k\left(v\right) \varepsilon_t^{+}\left(v-k\right) + \omega_k\left(v\right) \varepsilon_t^{-}\left(v-k\right)\right) h_t\left(v-k\right) + \sum_{j=1}^{p} \gamma_j\left(v\right) h_t^2\left(v-j\right). \end{aligned} \tag{1.5}$$

In (1.5), the notation $\varepsilon_t(\upsilon)$ refers to ε_t during the $\upsilon-th$ "season" $\upsilon \in \mathbb{S} = \{1,\ldots,s\}$ of cycle t, and, for convenience, we set $\varepsilon_t(\upsilon) = \varepsilon_{t-1}(\upsilon+s)$, $h_t(\upsilon) = h_{t-1}(\upsilon+s)$ and $e_t(\upsilon) = e_{t-1}(\upsilon+s)$ if $\upsilon < 0$. The non-peroidic notations (ϵ_t), (e_t) and (h_t) will be used interchangeably with the periodic one $(\varepsilon_t(\upsilon))$, $(e_t(\upsilon))$ and $(h_t(\upsilon))$ whenever emphasis on seasonality is not needed. It is worth noting that, since h_t^2 is the conditional variance of ϵ_t given the past information up to time $t-1$, the positivity of the functions $(\alpha_i(t), 0 \leqslant i \leqslant q)$, $(\beta_i(t), 1 \leqslant i \leqslant q)$ and $(\gamma_j(t), 1 \leqslant j \leqslant p)$ ensures the positivity of h_t^2 in $PTGARCH_s$ model. This is not the case in $PBLTGARCH_s$ even when $b_k(.) \geqslant 0$, $\omega_k(.) \geqslant 0$ and due to the penultimate term in (1.5), so the positivity of h_t^2 can be studied case by case and hence we shall assume throughout this paper that $h_t^2 > 0$, almost surely $(a.s)$.

Some algebraic notation and definitions are used throughout this paper. $O_{(n,m)}$ denotes the matrix of order $n \times m$ whose entries are zeros, for simplicity we set $O_{(n)} := O_{(n,n)}$ and $\underline{O}_{(n)} := O_{(n,1)}$. $I_{(n)}$ is the $n \times n$ identity matrix and $\mathbb{I}_\Delta$ denotes the indicator function of the set Δ. If $(M(i), i \in I)$ is $n \times n$ matrices sequence, we shall denote for any integer l and j, $\prod_{i=l}^{j} M(i) = M(l)M(l+1)\ldots M(j)$ if $l \leqslant j$ and $I_{(n)}$ otherwise. For any real random variable X, we denote $X^+ = \max(X,0)$, $X^- = \max(-X,0)$ so $X = X^+ - X^-$ and $|X| = X^+ - X^-$. $||.||$ refers to the induced norm in the space $\mathcal{M}(n,m)$ of $n \times m-$matrices. For instance, the norm of matrix $M = (m_{ij})$ is defined by $\|M\| = \sum |m_{ij}|$.

The main contributions of this paper can be summarized as follows. In Section 2, the Markovian representation of $PTBLGARCHs$ model is given and conditions for the existence of a strict periodic stationary (SPS) solution of (1.1)–(1.5) are established. In Section 3, the strong consistency and asymptotic normality of the $QMLE$ are studied. Numerical illustrations are given in Section 4 and an empirical application to the daily series of exchange rate of the Algerian Dinar against the single European currency is provided in Section 5.

2. Probabilistic properties of PBLTGARCHs(p,q,d)

As for many time series models, it is useful to write Equations (1.1)–(1.4) in an equivalent Markovian representation in order to facilitate their study. For this purpose, introduce the $r = (p+2q+2d)$-vector
$\underline{\varepsilon}_t' := \left(h_t^2,\ldots,h_{t-p+1}^2,\varepsilon_t^{+2},\varepsilon_t^{-2},\ldots,\varepsilon_{t-q+1}^{+2},\varepsilon_{t-q+1}^{-2},h_t e_t^+,h_t e_t^-,...,h_{t-d+1}e_{t-d+1}^+,h_{t-d+1}e_{t-d+1}^-\right)$
and $\underline{H}_0' := \left(1,\underline{O}_{(r-1)}'\right)$, $\underline{H}_1' := \left(\underline{O}_{(p)}',1,-1,\underline{O}_{(r-p-2)}'\right)$ and $\underline{\eta}_t(e_t) := \underline{\alpha}_{0,p+1}(t)\,e_t^{+2} + \underline{\alpha}_{0,p+2}(t)\,e_t^{-2} + \underline{\alpha}_{0,r-2d+1}(t)\,e_t^{+} + \underline{\alpha}_{0,r-2d+2}(t)\,e_t^{-} + \underline{\alpha}_{0,1}(t)$ in which the $j-th$ entry of $\underline{\alpha}_{0,j}(t)$ is $\alpha_0(t)$ and all other elements are 0. With these notations, we obtain the following state-space representation $\varepsilon_t^2 = \underline{H}_1'\underline{\varepsilon}_t$ and $h_t^2 = \underline{H}_0'\underline{\varepsilon}_t$

$$\underline{\varepsilon}_t = A_t(e_t)\,\underline{\varepsilon}_{t-1} + \underline{\eta}_t(e_t),\ t \in \mathbb{Z}, \tag{2.1}$$

with $A_t(e_t) := A_1(t)\,e_t^{+2} + A_2(t)\,e_t^{-2} + A_3(t)\,e_t^{+} + A_4(t)\,e_t^{-} + A_5(t)$. Here $(A_j(t), 1 \leqslant j \leqslant 5)$ are appropriate $(r \times r)$-periodic matrices easily obtained and uniquely determined by $\{\alpha_i(t),\ \beta_i(t),\ b_k(t),\ \omega_k(t),\ \gamma_j(t), 1 \leqslant i,k,j \leqslant q \vee p\}$. Now, by iterating (2.1) s times we get the following:

$$\underline{\varepsilon}_{(t+1)s} = H(\underline{e}_t)\,\underline{\varepsilon}_{ts} + \underline{\eta}(\underline{e}_t),\ t \in \mathbb{Z}, \tag{2.2}$$

where

$$\underline{e}_{t+1} = \left(e_{(t+1)s},...,e_{st+1}\right)', H(\underline{e}_t) = \left\{\prod_{j=0}^{s-1} A_{(t+1)s-j}\left(e_{(t+1)s-j}\right)\right\}, \underline{\eta}(\underline{e}_t) =$$
$$= \sum_{k=0}^{s-1}\left\{\prod_{j=0}^{s-1} A_{(t+1)s-j}\left(e_{(t+1)s-j}\right)\right\}\underline{\eta}_{(t+1)s-k}\left(e_{(t+1)s-k}\right).$$

Set $\underline{\varepsilon}_{ts} = \underline{\varepsilon}(t)$ (if there is no confusion). Then,(2.2) may be rewritten as

$$\underline{\varepsilon}(t) = H\left(\underline{e}_{t-1}\right)\underline{\varepsilon}(t-1) + \underline{\eta}\left(\underline{e}_{t-1}\right),\ t \in \mathbb{Z}. \tag{2.3}$$

Note here that $H\left(\underline{e}_t\right)$ is a sequence of $i.i.d.$ random matrices independent of $\underline{\varepsilon}(k), k \leqslant t$ and $\underline{\eta}\left(\underline{e}_t\right)$ is a sequence of $i.i.d.$ vectors. So, the existence of the so-called strictly periodically stationary (SPS) and periodic ergodic (PE) solutions to (1.1)–(1.5) is now equivalent to the existence of a strict stationary and ergodic solution to (2.3). Hence, equation similar to Equation (2.3) was examined by Bougerol and Picard [8] who established that the series

$$\underline{\varepsilon}(t) = \sum_{k\geqslant 1}\left\{\prod_{i=0}^{k-1} H\left(\underline{e}_{t-i-1}\right)\right\}\underline{\eta}\left(\underline{e}_{t-k-1}\right) + \underline{\eta}\left(\underline{e}_{t-1}\right), \tag{2.4}$$

constitute the unique, strictly stationary and ergodic solution of (2.3) if and only if, the top-Lyapunov exponent $\gamma(H)$ associated with the strictly stationary and ergodic sequence of random matrices $H = \left(H\left(\underline{e}_t\right), t \in \mathbb{Z}\right)$ defined by

$$\gamma(H) := \inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1} H\left(\underline{e}_{t-j-1}\right)\right\|\right\}\right\} \overset{a.s.}{=} \lim_{t\to\infty}\left\{\frac{1}{t}\log\left\|\prod_{j=0}^{t-1} H\left(\underline{e}_{t-j-1}\right),\right\|\right\} \tag{2.5}$$

is such that $\gamma(H) < 0$. However, the existence of $\gamma(H)$ is guaranteed by the fact that $E\{\log^+\|H(\underline{e}_t)\|\} \leqslant E\{\|H(\underline{e}_t)\|\} < \infty$, where $\log^+(x) = \max(\log x, 0)$ and the right-hand member in (2.5) can be justified using Kingman's [5] subadditive ergodic theorem. We summarize the above discussion in the following theorem due to Bougerol and Picard [8].

Theorem 2.1. *If $\gamma(H)$ corresponding to $PBLTGARCHs(q,d,p)$ models is strictly negative, then*
1. Equation (2.3) admits a unique, strictly stationary, causal and ergodic solution given by the series (2.4).
2. Equation (1.5) and, hence (1.1), admits a unique, SPS, causal and PE solution given by $h_t^2 = \underline{H}_0'\underline{\varepsilon}_t$ or $\varepsilon_t = e_t\left\{\underline{H}_1'\underline{\varepsilon}_t\right\}^{\frac{1}{2}}$ where $\underline{\varepsilon}_t$ is given by the series (2.4).

Proof. The proof follows essentially the same arguments as Bougerol and Picard [8]. □

Corollary 2.1. *If $\gamma(H) < 0$ and $E\{|e_0|^{2\delta}\} < \infty$ for some $\delta > 0$, then there is $\delta^* \in]0,1]$ such that $E(h_t^{\delta^*}) < \infty$ and $E({\varepsilon_t}^{\delta^*}) < \infty$.*

Remark 2.1. *Aknouche and Guerbyenne [2] have studied the conditions ensuring the existence and uniqueness of a SPS and PE solution of (1.1) and (1.5) using directly the (2.1) by showing that*

$$\inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{ts-1} A_{ts-j}\left(e_{ts-j}\right)\right\|\right\}\right\} \tag{2.6}$$

is a sufficient condition for that (2.1) to have a unique, causal, SPS and PE solution given by

$$\underline{\varepsilon}_t = \sum_{k\geqslant 1}\left\{\prod_{i=0}^{k-1} A_{t-i}\left(e_{t-i}\right)\right\}\underline{\eta}_{t-k}\left(e_{t-k}\right) + \underline{\eta}_t\left(e_t\right). \tag{2.7}$$

Remark 2.2. *It is worth noting that the condition $\gamma_L^{(s)}(H) < 0$ provides a certain global stability of model (2.1). However, when $\gamma_L^{(s)}(H) < 0$, the model (2.1) is said to be unstable and hence doesn't have a SPS solution. As an example, consider* *the* $PBLAARCH_s(1.1)$ *model defined by* $\varepsilon_t(\upsilon) = h_t(\upsilon)e_t(\upsilon)$ *and* $h_t^2(\upsilon) = \alpha_0(\upsilon) + \alpha_1(\upsilon)\left|e_t^2(\upsilon-1)\right|h_t^2(\upsilon-1) + b_1(\upsilon)|e_t(\upsilon-1)|h_t(\upsilon-1)$. *It is not difficult to show that* $\gamma_L^{(s)}(H) = E\left(\log\left(\prod_{\upsilon=0}^{s-1}\left(\left|\alpha_1(\upsilon)\left|e_0^2\right| + b_1(\upsilon)|e_0|\right|\right)\right)\right) \geqslant 0$. *Hence, the existence of some (not all) "stable regimes" (i.e.,* $E\left\{\log\left(\left|\alpha_1(\upsilon)\left|e_0^2\right| + b_1(\upsilon)|e_0|\right|\right)\right\} < 0$*) does not guarantee the existence of a SPS solution. More generally, we have the following convergence of the volatility to infinity for* $PBLAARCH_s(1,1)$ *process encompassing (2.2).*

Example 2.1. *In PBLTGARCHs(1; 1; 1) models, the necessary and sufficient condition ensuring the existence of strictly periodically stationary solution is that :*

$$\sum_{\upsilon=1}^{s} E\left\{\log\left\{\left|\alpha_1(\upsilon)e_0^{+2} + \beta_1(\upsilon)e_0^{-2} + b_1(\upsilon)e_0^{+} + \omega_1(\upsilon)e_0^{-} + \gamma(\upsilon)\right|\right\}\right\} < 0.$$

In particular, for standard $BLTARCH(1,1,1)$ *and for* $PBLTARCH_2(1,1,1)$ *with* $\alpha_1(1) = a$, $\omega_1(1) = b$, $\alpha_1(2) = 0.25a$, $\omega_1(2) = 0.25b$, $\beta_1(1) = \beta_1(2) = b_1(1) = b_1\ (2) = 0$ *and* $e_t \rightsquigarrow \mathcal{N}(0,1)$, *the stationarity zone is showed in Fig. 1.*

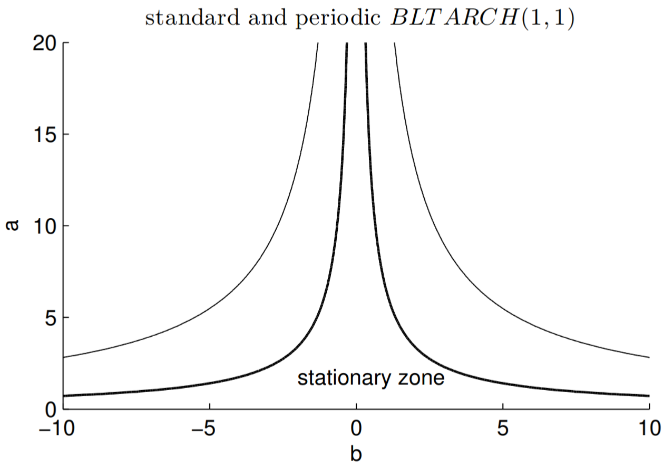

Fig. 1. Stationarity zones for standard (solid line) and periodic $BLTARCH(1,1)$ (dashed line)

It is clearly observed that the corresponding zone to the standard model is less restrictive than that corresponding to the periodic model.

2.1. Quasi-maximum likelihood estimator

In this subsection , we consider the quasi-maximum likelihood estimator ($QMLE$) for estimating the parameters of $PBLTGARCH_s$ model gathered in vector $\underline{\theta}' = \left(\underline{\theta}_1, \ldots, \underline{\theta}_{s(1+2q+2d+p)}\right) := \left(\underline{\alpha}', \underline{\beta}', \underline{b}', \underline{\omega}', \underline{\gamma}'\right) \in \Theta \subset \mathbb{R}^{s(1+2q+2d+p)}$, where $\underline{\alpha}' := \left(\underline{\alpha}_0', \underline{\alpha}_1', \ldots, \underline{\alpha}_q'\right)$, $\underline{\beta}' := \left(\underline{\beta}_1', \ldots, \underline{\beta}_q'\right)$, $\underline{b}' := \left(\underline{b}_1', \ldots, \underline{b}_d'\right)$, $\underline{\omega}' := \left(\underline{\omega}_1', \ldots, \underline{\omega}_d'\right)$, $\underline{\gamma}' := \left(\underline{\gamma}_1', \ldots, \underline{\gamma}_p'\right)$ with $\underline{\alpha}_i' := (\alpha_i(1), \ldots, \alpha_i(s))$, $\underline{\beta}_i' := (\beta_i(1), \ldots, \beta_i(s))$, $\underline{b}_k' := (b_k(1), \ldots, b_k(s))$ and $\underline{\omega}_k' := (\omega_k(1), \ldots, \omega_k(s))$, $\underline{\gamma}_j' := (\gamma_j(1), \ldots, \gamma_j(s))$ for all $0 \leqslant i \leqslant q$, $1 \leqslant k \leqslant d$ and $1 \leqslant j \leqslant p$. The true parameter value denoted by $\underline{\theta}_0 \in \Theta \subset \mathbb{R}^{s(1+2q+2d+p)}$ is unknown and, therefore, it must be estimated. For this purpose, consider a realization $\{\varepsilon_1, \ldots, \varepsilon_n; n = sN\}$ from the unique, causal, SPS and PE solution of (1.1) and (1.5) and let $h_t^2(\underline{\theta})$ be the conditional variance of ε_t given $\mathcal{F}_{t-1}$, where $\mathcal{F}_t := \sigma(\varepsilon_\tau; \tau \leqslant t)$. The Gaussian $log-$likelihood function of $\underline{\theta} \in \Theta$ conditional on some initial values $\varepsilon_0, \ldots, \varepsilon_{1-q}, h_0, \ldots, h_{1-p}$, which are generated by (1.1)–(1.5), is given up to an additive constant by $\tilde{L}_{Ns}(\underline{\theta}) = -(Ns)^{-1} \sum_{t=1}^{N} \sum_{v=0}^{s-1} \tilde{l}_{st+v}(\underline{\theta})$ with $\tilde{l}_t(\underline{\theta}) = \dfrac{\varepsilon_t^2}{\tilde{h}_t^2(\underline{\theta})} + \log \tilde{h}_t^2(\underline{\theta})$. Here $\widetilde{h}_t^2(\underline{\theta})$ is recursively defined, for $t \geqslant 1$ by $\tilde{h}_t^2(\underline{\theta}) = \alpha_0(t) + \sum_{i=1}^{q} \left(\alpha_i(t) \varepsilon_{t-i}^{+2} + \beta_i(t) \varepsilon_{t-i}^{-2}\right) + \sum_{k=1}^{d} \left(b_k(t) \varepsilon_{t-k}^{+} + \omega_k(t) \varepsilon_{t-k}^{-}\right) \tilde{h}_{t-k}(\underline{\theta}) + \sum_{j=1}^{p} \gamma_j(t) \tilde{h}_{t-j}^2(\underline{\theta})$. A $QMLE$ of $\underline{\theta}$ is defined as any measurable solution $\hat{\underline{\theta}}_{Ns}$ of $\hat{\underline{\theta}}_{Ns} = Arg \max_{\underline{\theta} \in \Theta} \tilde{L}_{Ns}(\underline{\theta}) = Arg \min_{\underline{\theta} \in \Theta} \left(-\tilde{L}_{Ns}(\underline{\theta})\right)$. In view of the strong dependency of $\tilde{h}_t^2(\underline{\theta})$ on initial values $\varepsilon_0, \ldots, \varepsilon_{1-q}, h_0, \ldots, h_{1-p}$, $\left(\tilde{l}_t(\underline{\theta})\right)_{t \geqslant 1}$ is neither a SPS nor a periodically ergodic (PE) process Therefore, it will be more convenient to work with an unobserved SPS and PE version. So, we work with an approximate version $\tilde{L}_{Ns} = -(NS)^{-1} \sum_{t=1}^{N} \sum_{v=0}^{s-1} l_{st+v}(\underline{\theta})$ of the likelihood $\tilde{L}_{Ns}(\underline{\theta})$ with $l_t(\underline{\theta}) = \dfrac{\varepsilon_t^2}{h_t^2(\underline{\theta})} + \log h_t^2(\underline{\theta})$.

3. Monte Carlo experiment

In this section, we describe the performance of the finite sample properties of the $QMLE$ of the unknown parameters in $BLTGARCH_s(1,1,1)$ model based on Monte Carlo experiments. To this end, we simulate $T = 500$ replications for different moderate sample sizes $n \in \{2000, 4000\}$ with standard $\mathcal{N}(0,1)$ and *student* $t_{(5)}$ as innovations distributions. The vector $\underline{\theta}$ of parameters is described in the bottom of each table below and is chosen to satisfy the strictly periodically stationary condition. All empirical results were obtained via implementation of our own scripts in $Matlab$ computing language. In the tables below, the columns correspond to the average of the parameters estimates over the N simulations. In order to show the performance of $QMLE$, the roots mean square error ($RMSE$) of the each $\hat{\theta}_n(i)$, $i = 1, \ldots, s$, (results between bracket), are reported in each table. Finally, the asymptotic distributions of $\hat{\theta}_n(v)$, $v = 1, \ldots, s$ over N simulations, followed by their boxplots summary, are plotted after each appropriate table.

3.1. Periodic BLTGARCH model

The example of our Monte Carlo experiment here is devoted to estimate the periodic $BLTGARCH_s(1,1,1)$ model with $s = 2$ according to standard $\mathcal{N}(0,1)$ and *student* $t_{(5)}$ as innovations distributions. The vector of parameters to be estimated is thus $\underline{\theta} = (\underline{\alpha}_0', \underline{\alpha}_1', \underline{\beta}_1', \underline{b}_1', \underline{\omega}_1', \underline{\gamma}_1')'$ where $\underline{\alpha}_0' = (\alpha_0(1), \alpha_0(2))$, $\underline{\alpha}_1' = (\alpha_1(1), \alpha_1(2))'$, etc. . . are subjected to two models Model (1) and Model (2) described as: Model(1): The parameters are chosen to ensure the locally strictly stationarity condition i.e., for each $v = 1, 2$, $E\left\{\log\left|\alpha_1(v) e_0^{+2} + \beta_1(v) e_0^{-2} + b_1(v) e_0^{+} + \omega_1(v) e_0^{-} + \gamma_1(v)\right|\right\} < 0$, so $\left(h_t^2\right)_t$ is strict periodic stationary. Model (2): The parameters are chosen such that $E\left\{\log\left|\alpha_1(1) e_0^{+2} + \beta_1(1) e_0^{-2} + b_1(1) e_0^{+} + \omega_1(1) e_0^{-} + \gamma_1(1)\right|\right\} > 0$, but $\sum_{v=1}^{2} E\left\{\log\left|\alpha_1(v) e_0^{+2} + \beta_1(v) e_0^{-2} + b_1(v) e_0^{+} + \omega_1(v) e_0^{-} + \gamma_1(v)\right|\right\} < 0$, to ensure the strict periodic stationarity condition of $\left(h_t^2\right)_t$. The results of simulation according to both models (1) and (2) are given in Tab. 1.

The asymptotic distribution of the sequence $\left(\sqrt{n}\left(\hat{\underline{\theta}}_n(i) - \underline{\theta}(i)\right)\right)_{n \geqslant 1}$, $i = 1, \ldots, 12$ followed by their boxplot summary according to model(1) of Tab. 1 are shown in Fig. 2.

Table 1. Average and $RMSE$ of 500 simulations of $QMLE$ for $PBLTGARCH_2(1,1,1)$

		$\mathcal{N}(0,1)$		$t_{(5)}$	
n	υ	2000	4000	2000	4000
$\hat{\underline{\alpha}}_0$	1	0.9888 (0.0264)	0.9953 (0.0133)	0.9561 (0.0739)	0.9921 (0.0280)
	2	0.9944 (0.0286)	0.9928 (0.0134)	0.9515 (0.0728)	0.9721 (0.0343)
$\hat{\underline{\alpha}}_1$	1	0.4999 (0.0335)	0.4947 (0.0162)	0.5048 (0.0918)	0.4971 (0.0427)
	2	0.5008 (0.0414)	0.4973 (0.0203)	0.4905 (0.0944)	0.5038 (0.0598)
$\hat{\underline{\beta}}_1$	1	0.3631 (0.0468)	0.3562 (0.0242)	0.3661 (0.1020)	0.3596 (0.0608)
	2	0.3359 (0.0324)	0.3411 (0.0154)	0.3464 (0.0725)	0.3468 (0.0352)
$\hat{\underline{b}}_1$	1	−0.2607 (0.0688)	−0.2473 (0.0333)	−0.2760 (0.1567)	−0.2482 (0.0940)
	2	−0.0027 (0.0805)	0.0058 (0.0392)	0.0084 (0.1868)	−0.0054 (0.1054)
$\hat{\underline{\omega}}_1$	1	0.3240 (0.0977)	0.3412 (0.0500)	0.3198 (0.2002)	0.3459 (0.1247)
	2	0.0126 (0.0720)	0.0087 (0.0348)	−0.0030 (0.1636)	−0.0093 (0.0814)
$\hat{\underline{\gamma}}_1$	1	0.1598 (0.0093)	0.1527 (0.0043)	0.1793 (0.0280)	0.1549 (0.0110)
	2	0.1578 (0.0090)	0.1530 (0.0044)	0.1807 (0.0284)	0.1680 (0.0124)

Model(1) : $\underline{\theta} = (1.00, 1.00, 0.50, 0.50, 0.35, 0.35, -0.25, 0.00, 0.35, 0.00, 0.15, 0.15)'$

$\hat{\underline{\alpha}}_0$	1	0.9844 (0.0554)	0.9935 (0.0242)	0.9821 (0.1133)	0.9933 (0.0586)
	2	1.0391 (0.1631)	1.0152 (0.0764)	0.9648 (0.3207)	0.9962 (0.1817)
$\hat{\underline{\alpha}}_1$	1	0.5023 (0.0231)	0.4959 (0.0119)	0.5112 (0.0662)	0.5023 (0.0350)
	2	0.4819 (0.0640)	0.4870 (0.0327)	0.5140 (0.1545)	0.5203 (0.0872)
$\hat{\underline{\beta}}_1$	1	0.2571 (0.0108)	0.2541 (0.0055)	0.2757 (0.0321)	0.2512 (0.0148)
	2	0.4179 (0.0600)	0.4324 (0.0305)	0.4290 (0.1037)	0.4434 (0.0883)
$\hat{\underline{b}}_1$	1	0.2459 (0.0324)	0.2550 (0.0166)	0.2316 (0.0893)	0.2544 (0.0496)
	2	0.1794 (0.1533)	0.1748 (0.0765)	0.1381 (0.3424)	0.1219 (0.1929)
$\hat{\underline{\omega}}_1$	1	0.1404 (0.0204)	0.1470 (0.0103)	0.1175 (0.0569)	0.1495 (0.0261)
	2	0.1905 (0.1395)	0.1739 (0.0759)	0.1477 (0.2891)	0.1490 (0.1913)
$\hat{\underline{\gamma}}_1$	1	0.1532 (0.0018)	0.1502 (0.0008)	0.1587 (0.0061)	0.1518 (0.0032)
	2	0.7415 (0.0185)	0.7419 (0.0098)	0.7734 (0.0532)	0.7584 (0.0315)

Model(2) : $\underline{\theta} = (1.00, 1.00, 0.50, 0.50, 0.25, 0.45, 0.25, 0.15, 0.15, 0.15, 0.15, 0.75)'$

Comments: A quick glance to the results of Monte Carlo experiment shows that the results of Tab. 1 provide the parameters estimates of $PBLTGARCH_s(1,1,1)$, with $s = 2$ fitted on Model (1) and Model (2) generated by standard $\mathcal{N}(0,1)$ and *student* $t_{(5)}$ innovations through 500 independent simulations. First, it is clear that the results of QML associated with $t_{(5)}$ innovations have a poor performance compared with those associated to $\mathcal{N}(0,1)$. In general, it can be observed that the parameters associated to these models are quite well estimated with non significant deviations in estimated values for two innovations errors $\mathcal{N}(0,1)$ and $t_{(5)}$. It is worth noting that some values of estimates have a moderate standard deviation. In Tab. 1 where two models was simulated following a $PBLTGARCH_s(1,1,1)$ model in which the parameters of the two regimes in Model(1) are such that $E\left\{\log\left|\alpha_1(\upsilon)\, e_0^{+2} + \beta_1(\upsilon)\, e_0^{-2} + b_1(\upsilon)\, e_0^{+} + \omega_1(\upsilon)\, e_0^{-} + \gamma_1(\upsilon)\right|\right\} < 0$, $\upsilon = 1, \ldots, 2$, whereas, in Model(2) the second regime is explosive in the sense that $E\left\{\log\left|\alpha_1(2)\, e_0^{+2} + \beta_1(2)\, e_0^{-2} + b_1(2)\, e_0^{+} + \omega_1(2)\, e_0^{-} + \gamma_1(2)\right|\right\} > 0$, but the SPS of the model is ensured. Also, one can see that the results reveal in general quite satisfactory in accordance with the asymptotic theory results.

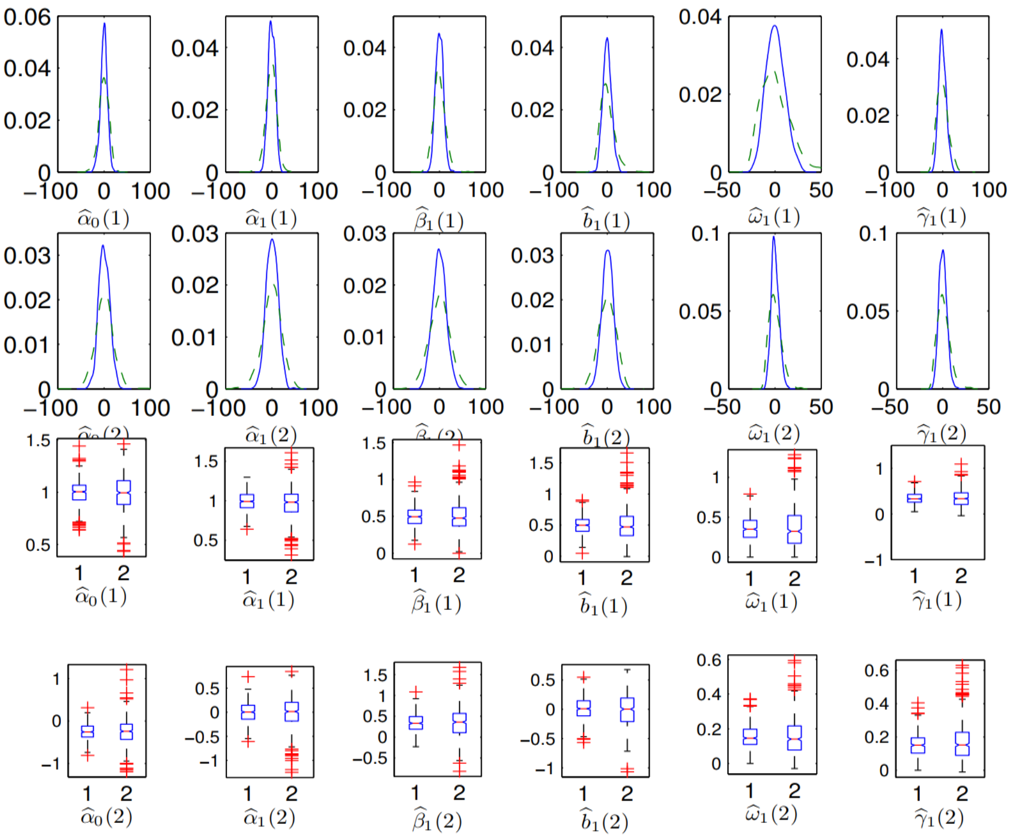

Fig. 2. Top panels: the asymptotic distribution of $\sqrt{n}(\hat{\underline{\theta}}_n(i) - \underline{\theta}(i))$ (full line for *Normal* and dashed line for *Student*). Bottom panels: Box plot summary of $\hat{\underline{\theta}}_n(i)$, $i = 1, \ldots, 12$ (1 for *Normal* and 2 for *Student*) according to Model(1) of Tab. 1

4. Applications on exchange rates

The proposed model is investigated with real financial time series. So, we apply our model for modelling the foreign exchange rates of Algerian Dinar against the European currency (*Euro*) denoted by y_t already analyzed by Hamdi and Souam [3] via a mixture periodic *GARCH* models. We consider returns series $(r_t = 100 \times (\log(y_t/y_{t-1})))_{t\geqslant 1}$ of daily exchange rates of Algerian Dinar against Euro. The observation covers the period from January 3, 2000 to September 29, 2011. Since some weeks comprise less than five observations (due to legal holidays), we remove the entire weeks with less than five data available rather than estimating the "pseudo-missing" observations by an ad-hoc method. Thus, the final length of transformed data is 3055 observations uniformly distributed on 611 weeks. Fig. 3 displays the plots of the series (y_t) and its returns (r_t), squared return (r_t^2) and absolute return $(|r_t|)$.

By quickly examining the plots in Fig. 3, we can see that the original series are non stationary (since these do not fluctuate around a constant mean) and non-linear contrary to their returns that appear to be stationry. Moreover, there is no clear discernible behavior pattern in the returns, but some persistence is indicated in the plots of the squared and absolute returns. Additionally, some elementary statistics of the series $(y_t)_{t\geqslant 1}$ and its returns $(r_t)_{t\geqslant 1}$, squared return $(r_t^2)_{t\geqslant 1}$ and absolute return $(|r_t|)_{t\geqslant 1}$ are displeyed in Tab. 2

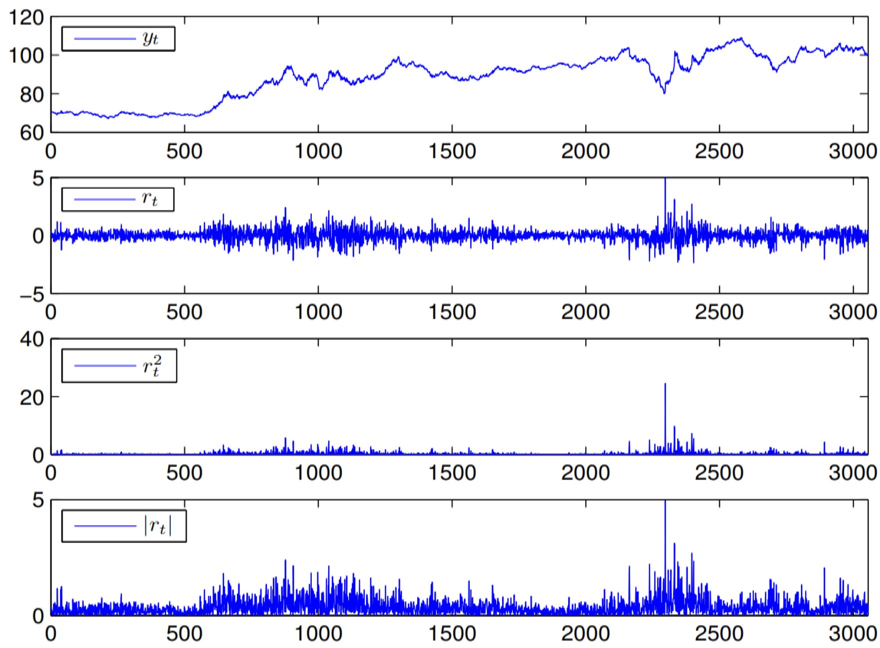

Fig. 3. The plots of the series(y_t) , squared r_t and absolute (r_t)

Table 2. Elementary statistics of the series $(y_t)_{t\geqslant 1}$, $(r_t)_{t\geqslant 1}$, $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$

Series	Means	Std.Dev	Median	Skewness	Kurtosis
y_t	88.6118	11.5755	91.0995	−0.5181	2.1330
r_t	0.0118	0.5043	0.0123	0.3536	8.9678
r_t^2	0.2543	0.7193	0.0652	16.1027	464.3694
$\|r_t\|$	0.3575	0.3557	0.2554	2.6956	18.4307

Tab. 2 presents statistical summary of the series $(y_t)_{t\geqslant 1}$, $(r_t)_{t\geqslant 1}$, $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$ with summary measures of normality test results. The return $(r_t)_{t\geqslant 1}$ exhibits non-zero skewness and leptokurtic, while $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$ exhibit significant skewness and kurtosis, indicating that their distribution is more peaked with a thicker tails than the normal distribution. Fig. 4 displays the sample autocorrelations functions (ACF) of the series $(r_t)_{t\geqslant 1}$, $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$ computed at 40 lags.

In Fig. 4, we can see that the log returns $(r_t)_{t\geqslant 1}$ show no evidence of serial correlation, but the squared and absolute returns are positively autocorrelated. Also, the decay rates of the sample autocorrelations of $\left(r_t^2\right)_{t\geqslant 1}$ and $(|r_t|)_{t\geqslant 1}$ appear to be violated compared with the correlation associated to an $ARMA$ process suggesting possibly a non linear behavior for modelling purpose.

4.1. Modeling with standard BLTGARCH model

The first attempt will be modeling the series $(r_t)_{t\geqslant 1}$ by a standard $BLTGARCH(1,1,1)$ model. The parameters estimates of volatility $(\hat{h}_t^{(s)})_{t\geqslant 1}$ to $BLTGARCH(1,1,1)$ with their $RMSE$ are given in Tab. 3.

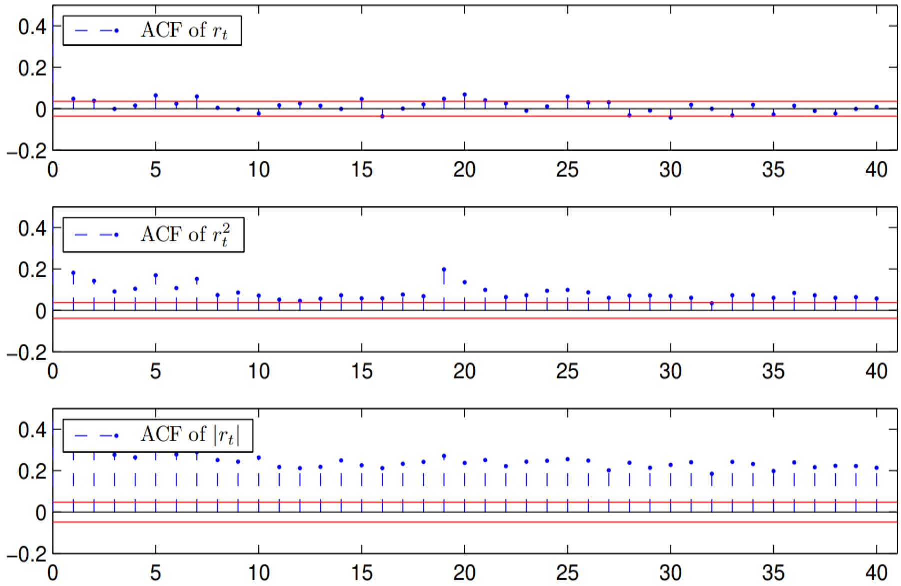

Fig. 4. The ACF of the returns and of their squred and absolute series

Table 3. Parameters estimates and their $RMSE$ of the volatilities $(\hat{h}_t^{(s)})_{t\geqslant 1}$

Parameters	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{b}_1$	$\hat{\omega}_1$	$\hat{\gamma}_1$
$\left(\hat{h}_t^{(s)}\right)_{t\geqslant 1}$	0.0007 (0.0005)	0.0304 (0.0176)	0.0591 (0.0224)	0.0276 (0.0439)	0.0283 (0.0430)	0.9540 (0.0175)

The plot of the estimated volatility $(\hat{h}_t^{(s)})_{t\geqslant 1}$ is shown later in the left side of Fig. 5.

4.2. Modeling with PBLTGARCH model

The second attempt is to look for a model able to cover the day-of -week seasonality in return (r_t) (see for instance Franses and Raap [9]). So, in order to analyze the seasonality, we fitted the following simple $PBLTGARCH_5(1,1,1)$ model for each series or equivalently. Hence, we estimate its volatility process $(h_t^2)_{t\geqslant 1}$ through five periodic effects, $r_t = h_t e_t$ and

$$h_t^2 = \alpha_0\,(t) + \left(\alpha_1\,(t)\,r_{t-1}^{+2} + \beta_1\,(t)\,r_{t-1}^{-2}\right) + \left(b_1\,(t)\,r_{t-1}^{+} + \omega_1\,(t)\,r_{t-1}^{-}\right)h_{t-k} + \gamma_1\,(t)\,h_{t-1}^2. \quad (14)$$

The parameters estimates of five-regimes (intra-day) of $(\hat{h}_t^{(p)})_{t\geqslant 1}$ and their $RMSE$ according to model (14) are reported in Tab. 4.
The plots of estimated volatilities and the squared returns associated to ($Euro$) are showed in Fig. 5.

4.3. Comments

Tab. 3 and Tab. 4 display the $(\hat{h}_t)_{t\geqslant 1}$ estimated by Standard $BLTGARCH\,(1,1,1)$ and Periodic $BLTGARCH_5\,(1,1,1)$ models and reflect some characteristics of "spurious" $GARCH$ effects. In particular, the components of $\underline{\hat{\alpha}}_0$ are close to zeros while the components of $\underline{\hat{\gamma}}_1$ are close

Table 4. Parameters estimates and their $RMSE$ of the volatilities $(\hat{h}_t^{(p)})$

days	$\underline{\hat{\alpha}}_0$	$\underline{\hat{\alpha}}_1$	$\underline{\hat{\beta}}_1$	$\underline{\hat{b}}_1$	$\underline{\hat{\omega}}_1$	$\underline{\hat{\gamma}}_1$
Sunday	0.0001	0.0145	0.0032	0.0165	0.0520	1.1826
	(0.0320)	(0.0329)	(0.0812)	(0.0926)	(0.1234)	(0.1894)
Monday	0.0010	0.0082	0.0419	0.0685	0.0831	1.0009
	(0.0296)	(0.0563)	(0.0588)	(0.2913)	(0.1429)	(0.1326)
Tuesday	0.0001	0.0015	0.0376	0.1162	0.0318	0.8504
	(0.0289)	(0.0651)	(0.0171)	(0.0611)	(0.0662)	(0.1156)
Wednesday	0.0025	0.0869	0.0648	0.0659	0.1768	0.7941
	(0.0142)	(0.0322)	(0.0345)	(0.1136)	(0.0951)	(0.0955)
Thursday	0.0002	0.0082	0.0645	0.0909	0.0229	0.9803
	(0.0160)	(0.0799)	(0.1260)	(0.2751)	(0.3544)	(0.2810)

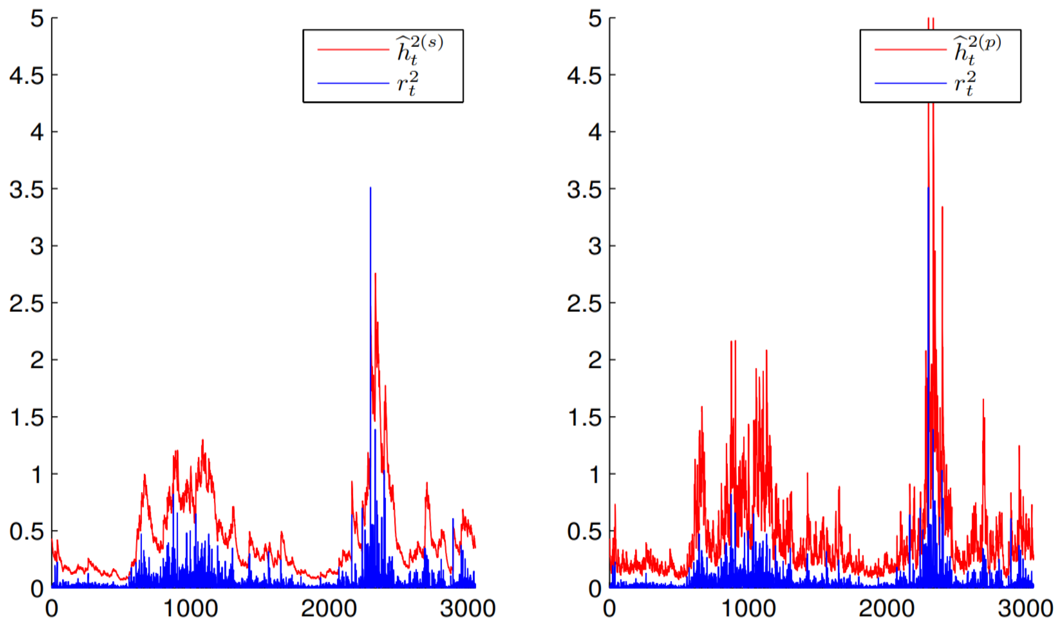

Fig. 5. Dark blue: squared returns, light red: volatilities estimates according to Standard $BLTGARCH(1,1,1)$ (left) and to Periodic $BLTGARCH_5(1,1,1)$ (right)

to ones with moderate $RMSE$. Fig. 5 represents the plots of the volatilities estimates (plots in red) according to $BLTGARCH\,(1,1,1)$ model (left) and $PBLTGARCH_5\,(1,1,1)$ model (right) and compared with the appropriate squared returns (plots in blue). It also demonstrates that a large piece of returns (positive or negative) leads to a high volatility and a small piece of returns leads to a low volatility, indicating volatility clustering. In particular, the period between 2000 and 2002 is characterized by low volatility levels compared to the period between 2009 and 2010 for both series. In addition, a high volatility cluster beginning in 2005 is observed and is mainly due to the global financial crisis. After this period of uncertainty, a cluster of low volatility is observed during 3 years. An other high volatility cluster is detected and could be related to the devaluation of the Dinar. Finally, the conditional volatility seems to be more stable after 2010. Our empirical results demonstrate that it is very difficult to distinguish between the volatilities $(\hat{h}_t^{(s)})_{t\geqslant 1}$ and $(\hat{h}_t^{(p)})_{t\geqslant 1}$ in Fig. 5, except perhaps, that the volatilities $(\hat{h}_t^{(p)})_{t\geqslant 1}$ is more fluctuated than $(\hat{h}_t^{(s)})_{t\geqslant 1}$. This finding may indicate the presence of a certain (hidden) periodicity in $(\hat{h}_t^{(p)})_{t\geqslant 1}$.

Conclusion

Beside the probabilistic structure and the conditions ensuring the existence of a SPS solution, this paper studies also the asymptotic properties of the quasi-maximum likelihood estimators of $PBLTGARCH(q, d, p)$ model. Indeed, for the first part, we have given the necessary and sufficient conditions for the existence of a strictly periodically stationary solution based on the negativity of the top-Lyapunov exponent. The paper presents for the second part, the theoretical results, which are illustrated in the third part by a Monte Carlo experiment through some usual innovations and an application to the exchange rate of the Algerian Dinar against the Euro showing its performance and its efficiency.

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О периодических билинейных пороговых моделях GARCH

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Аннотация. Периодические обобщенные авторегрессионные условно гетероскедастические модели ($PGARCH$) были представлены Bollerslev et Ghysels. Эти модели вызвали значительный интерес и продолжают привлекать внимание исследователей. Данная статья посвящена расширению стандартной билинейной пороговой модели $GARCH$ ($BLTGARCH$) до модели с периодически меняющимися во времени коэффициентами ($PBLTGARCH$). В этом классе моделей допускается переключение параметров между разными режимами. Более того, эти модели позволяют интегрировать асимметричные эффекты волатильности. Во-первых, мы приводим необходимые и достаточные условия, обеспечивающие существование стационарных решений (в периодическом смысле). Во-вторых, разработан подход оценки квазимаксимального правдоподобия (QML) для оценки модели $PBLTGARCH$. Точнее, сильная состоятельность и асимптотическая нормальность оценки изучаются при мягких условиях регулярности, требующих строгой стационарности и конечности моментов некоторого порядка для члена ошибки. Свойства $QMLE$ для конечной выборки иллюстрируются исследованием Монте-Карло. Наконец, предложенная нами модель применяется для моделирования обменных курсов алжирского динара по отношению к единой европейской валюте (*Euro*).

Ключевые слова: периодические билинейные пороговые модели $GARCH$, строго периодически стационарная, гауссовская оценка QML.