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On the Real Roots of Systems of Transcendental Equations

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Abstract. The work is devoted to finding the number of real roots of systems of transcendental equations. It is shown that if a system has simple roots, then the number of real coordinates of the roots is the same. Therefore, the number of real roots is related with the number of real roots of the resultant of the system.

Keywords: system of transcendental equations, resultant, residue integral.

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Introduction

Finding the number of real roots of polynomials with real coefficients is a classical problem of algebra. There are quite a few related results; the Hermite method of quadratic forms [1, ch. 16, Sec. 9], [2, Appendix I], Sturm's theorem, Descartes' sign rule, the Budan–Fourier theorem (see, for example, [3, Chapter 9]). Further development of these methods for polynomials can be found in the paper by M. Krein and M. Naimark [4] (in fact, this paper was published in 1936 in Russian, but has long become a bibliographic rarity) and the monograph by Jury [5]. For entire functions, the question of localization of real positive roots was considered in the classical works of N. G. Chebotarev [6, p. 3–18, 29-56], as well as in the work [7] (we refer to the collected works of N. G. Chebotarev, since his original works are now inaccessible).

For systems of equations, the number of real roots was studied in the articles [8–10]. In the article [11] root coordinates were related to the root first coordinates.

The monographs [12,13] consider algebraic and transcendental systems of equations. Systems of transcendental equations arise, for example, when studying the equations of chemical kinetics [14]. One of the problems that arises there is the problem of the number of real positive roots of a system of equations, or the number of roots in the reaction polyhedron.

1. Resultant of a system

Consider a system of equations of the form

$$\begin{cases} f_1(z) = 0, \\ \dots \\ f_n(z) = 0, \end{cases} \tag{1}$$

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where $f_1(z), \ldots, f_n(z)$ are entire functions of complex variables $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$. In what follows we will assume that the set of roots of the system (1) is discrete. Therefore it is no more than countable. Let $\mathcal{E}$ denote the set of roots with non-zero coordinates $w_{(\nu)} = \big(w_{1(\nu)}, \ldots, w_{n(\nu)}\big)$, $\nu = 1, 2, \ldots$, numbered in ascending order of modules: $\left|w_{(1)}\right| \leqslant \left|w_{(2)}\right| \leqslant \ldots \leqslant \left|w_{(\nu)}\right| \leqslant \ldots$.

Let us consider power sums of roots S_α, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a non-negative multi-index (all components are non-negative and integer) and $\alpha_1 + \ldots + \alpha_n > 0$ of the form

$$S_\alpha = \sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}^{\alpha_1} \cdot w_{2(\nu)}^{\alpha_2} \cdots w_{n(\nu)}^{\alpha_n}}.$$

We will assume that all series S_α are absolutely convergent for any multi-indices α.

The concept of power sums (in the negative power) for transcendental systems of equations was considered in the works [15–18]. The results of these papers were based on the calculation of power sums through the so-called residue integrals [19].

Lemma 1. *The series S_α converge absolutely for any multi-indices α if and only if the series*

$$\sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}}, \quad \ldots, \quad \sum_{\nu=1}^{\infty} \frac{1}{w_{n(\nu)}}$$

converge absolutely.

Proof see [11].

Therefore, an entire function of genus zero is defined ([20], Chapter 7)

$$R(z_1) = z_1^s \cdot \prod_{\eta=1}^{\infty} \left(1 - \frac{z_1}{w_{1(\eta)}}\right), \tag{2}$$

where s is the multiplicity of the zero of the system (1) at the zero point, $s \geqslant 0$.

In the formula (2), the infinite product converges absolutely and uniformly in the complex plane $\mathbb{C}$.

We will call the function $R(z_1)$ the *resultant* of the system (1) with respect to the variable z_1. The concept of a resultant for systems of transcendental equations is not generally accepted. For the case of two equations it was introduced by N. G. Chebotarev [6] (p. 18–27). In recent years, this concept has been considered in the works of [21–24]. The results of these papers were based on the calculation of power sums through the so-called residue integrals [19]. The main problem is to find coefficients of a resultant without knowing the roots themselves. In this sense, determining the resultant for a system is not constructive. There is no formula for systems of equations like there is for the Sylvester resultant for polynomials. Some approaches to finding it can be found in the monograph [12, Sec. 3.7].

2. Auxiliary results

Let us introduce the functions $P_j^{(t)}(z_1)$

$$P_j^{(t)}(z_1) = -z_1^{s-1} \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{j(\nu)}^t} \cdot \prod_{\eta \neq \nu} \left(1 - \frac{z_1}{w_{1(\eta)}}\right), \quad t \geqslant 0, \quad s \geqslant 1. \tag{3}$$

Lemma 2. *Functions (3) are entire functions of the variable z_1.*

Proof. Let us write $P_j^{(t)}(z_1)$ in the form:

$$P_j^{(t)}(z_1) = -z_1^{s-1} \cdot \prod_{\eta=1}^{\infty} \left(1 - \frac{z_1}{w_{1(\eta)}}\right) \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{j(\nu)}^t} \cdot \frac{1}{1 - \frac{z_1}{w_{1(\nu)}}}.$$

The infinite product $\prod\limits_{\eta=1}^{\infty} \left(1 - \dfrac{z_1}{w_{1(\eta)}}\right)$ is an entire function of genus zero.

Let us prove that the series

$$\sum_{\nu=1}^{\infty} \frac{1}{w_{j(\nu)}^t} \cdot \frac{1}{1 - \frac{z_1}{w_{1(\nu)}}}$$

converges absolutely and uniformly in the complex plane $\mathbb{C}$.

By Lemma 1 the series $\sum\limits_{\nu=1}^{\infty} \dfrac{1}{|w_{1(\nu)}|}$ converges, then

$$\lim_{\nu\to\infty} \frac{1}{|w_{1(\nu)}|} = 0,$$

and therefore

$$\lim_{\nu\to\infty} \left(1 - \frac{z_1}{w_{1(\nu)}}\right) = 1.$$

Since $1 - \dfrac{z_1}{w_{1(\nu)}}$ is close to unity, we can assume that

$$\sum_{\nu=1}^{\infty} \left| \frac{1}{w_{j(\nu)}^t} \cdot \frac{1}{1 - \frac{z_1}{w_{1(\nu)}}} \right| \leqslant 2 \cdot \sum_{\nu=1}^{\infty} \frac{1}{\left|w_{j(\nu)}^t\right|}.$$

Whence it follows that the series $\sum\limits_{\nu=1}^{\infty} \dfrac{1}{w_{j(\nu)}^t} \cdot \dfrac{1}{1 - \frac{z_1}{w_{1(\nu)}}}$ converges absolutely and uniformly in the complex plane $\mathbb{C}$.

This proves that the functions $P_j^{(t)}(z_1)$ are entire functions of the variable z_1. □

Theorem 1. *Let the function $R(z_1)$ have simple zeros $w_{1(\nu)}$, $\nu = 1, 2, \ldots$. Then the next equality is true*

$$\left.\frac{P_j^{(t)}(z_1)}{R'(z_1)}\right|_{z_1 = w_{1(\mu)}} = \frac{1}{w_{j(\mu)}^t} \quad \text{for any} \quad \mu.$$

Proof. Let us find the derivative with respect to z_1 of the function $R(z_1)$:

$$R'(z_1) = s \cdot z_1^{s-1} \cdot \prod_{\eta=1}^{\infty} \left(1 - \frac{z_1}{w_{1(\eta)}}\right) - z_1^s \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}} \cdot \prod_{\eta\neq\nu} \left(1 - \frac{z_1}{w_{1(\eta)}}\right).$$

The first term calculated at the point $z_1 = w_{1(\mu)}$ is equal to 0, since

$$\left.\prod_{\eta=1}^{\infty} \left(1 - \frac{z_1}{w_{1(\eta)}}\right)\right|_{z_1 = w_{1(\mu)}} = 0.$$

Let us calculate the second term at the point $z_1 = w_{1(\mu)}$:

$$-w_{1(\mu)}^s \cdot \sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}} \cdot \prod_{\eta\neq\nu} \left(1 - \frac{w_{1(\mu)}}{w_{1(\eta)}}\right) = -w_{1(\mu)}^s \cdot \frac{1}{w_{1(\mu)}} \cdot \prod_{\eta\neq\mu} \left(1 - \frac{w_{1(\mu)}}{w_{1(\eta)}}\right).$$

Thus,

$$R^{'}(z_1)\Big|_{z_1=w_{1(\mu)}} = -w_{1(\mu)}^{s-1} \cdot \prod_{\eta\neq\mu}\left(1-\frac{w_{1(\mu)}}{w_{1(\eta)}}\right).$$

Let us find the value of $P_j^{(t)}(z_1)$ at the point $z_1 = w_{1(\mu)}$:

$$\begin{aligned} P_j^{(t)}(z_1)\Big|_{z_1=w_{1(\mu)}} &= -w_{1(\mu)}^{s-1} \cdot \sum_{\nu=1}^{\infty}\frac{1}{w_{j(\nu)}^t} \cdot \prod_{\eta\neq\nu}\left(1-\frac{w_{1(\mu)}}{w_{1(\eta)}}\right) = \\ &= -w_{1(\mu)}^{s-1} \cdot \frac{1}{w_{j(\mu)}^t} \cdot \prod_{\eta\neq\nu}\left(1-\frac{w_{1(\mu)}}{w_{1(\eta)}}\right). \end{aligned}$$

After substituting the found expressions into $\left.\dfrac{P_j^{(t)}(z_1)}{R^{'}(z_1)}\right|_{z_1=w_{1(\mu)}}$ and reducing it, we obtain the statement of the theorem. □

Thus, we get that if the first coordinates of the roots from $\mathcal{E}$ are known, then to find the remaining coordinates of the roots there is no need to find resultants for other variables.

As a resultant of the system (1), we can also take a function of the form

$$Q(z_1) = z_1^s \cdot e^{g(z_1)} \cdot \prod_{\eta=1}^{\infty}\left(1-\frac{z_1}{w_{1(\eta)}}\right). \tag{4}$$

where $g(z_1)$ is some entire function, s is the multiplicity of the zero of the system (1) at zero, $s \geqslant 0$.

It has the same roots and the same multiplicity as the resultant $R(z_1)$.

Consider the system of functions

$$V_j^{(t)}(z_1) = -z^{s-1} \cdot e^{g(z_1)} \cdot \sum_{\nu=1}^{\infty}\frac{1}{w_{j(\nu)}^t} \cdot \prod_{\eta\neq\nu}\left(1-\frac{z_1}{w_{1(\eta)}}\right), \quad t \geqslant 1, \quad s \geqslant 1.$$

Corollary 1. *Let the function $Q(z_1)$ have simple zeros $w_{1(\nu)}$, $\nu = 1, 2, \ldots$. Then the next equality is true*

$$\left.\frac{V_j^{(t)}(z_1)}{Q'(z_1)}\right|_{z_1=w_{1(\mu)}} = \frac{1}{w_{j(\mu)}^t}.$$

The proof of Corollary 1 repeats the proof of Theorem 1.

3. Main results

Let us write the Taylor series expansion in the variable z_1 in the neighborhood of zero of the function $P_j^{(t)}(z_1)$ and the function $R(z_1)$:

$$P_j^{(t)}(z_1) = -z_1^{s-1} \cdot \sum_{m=0}^{\infty} a_{jm}^{(t)} \cdot z_1^m, \qquad a_{j0}^{(t)} = 1,$$

$$R(z_1) = z_1^s \cdot \sum_{m=0}^{\infty} b_m \cdot z_1^m, \qquad b_0 = 1,$$

Theorem 2. *If a system (1) with real coefficients is such that all zeros of $R(z_1)$ are simple except for the point $z_1 = 0$, then the number of real roots of the system (1) in $\mathcal{E}$ coincides with the number of real roots of the resultant $R(z_1)$.*

Proof. If the system (1) has real coefficients, then all power sums of the roots S_α are real.

Indeed, let the system (1) has a real root w, that is, $f_j(w) = 0$, $j = 1, \dots, n$. Then $\overline{f_j(w)} = 0$, $j = 1, \dots, n$. Since the system (1) has real coefficients, then $f_j(\overline{w}) = 0$, $j = 1, \dots, n$. Therefore, $\overline{w}$ is also a root. That is, complex roots are paired. This means that in the power sum S_α each non-real (complex) term corresponds to a complex conjugate term. And therefore the sum of these numbers is a real number.

Let us prove that the resultant $R(z_1) = z_1^s \cdot \sum\limits_{m=0}^{\infty} b_m \cdot z_1^m$ has real coefficients, that is, that b_m are real , $m = 0, 1, 2, \dots$.

To do this, consider the infinite product

$$\prod_{\eta=1}^{\infty}\left(1 - \frac{z_1}{w_{1(\eta)}}\right) = 1 + z_1 \cdot \sum_{j=1}^{\infty}\frac{-1}{w_{1(j)}} + z_1^2 \cdot \sum_{j_1<j_2}\frac{1}{w_{1(j_1)} \cdot w_{1(j_2)}} +$$

$$+ z_1^3 \cdot \sum_{j_1<j_2<j_3}\frac{-1}{w_{1(j_1)} \cdot w_{1(j_2)} \cdot w_{1(j_3)}} + \ldots =$$

$$= 1 + \sum_{m=1}^{\infty}(-1)^m \cdot z_1^m \cdot \sum_{j_1<j_2<\ldots<j_m}\frac{1}{w_{1(j_1)} \cdot w_{1(j_2)} \cdot \ldots \cdot w_{1(j_m)}}.$$

The coefficients for z_1^m are equal to:

$$\begin{aligned} b_0 &= 1, \\ b_m &= (-1)^m \cdot \sum_{j_1<j_2<\ldots<j_m}\frac{1}{w_{1(j_1)} \cdot w_{1(j_2)} \cdot \ldots \cdot w_{1(j_m)}}, \quad m = 1, 2, \ldots \end{aligned} \tag{5}$$

From the form (5) it obviously follows that b_m are symmetric functions of the numbers $\frac{1}{w_{1(1)}}$, $\frac{1}{w_{1(2)}}$, $\frac{1}{w_{1(3)}}$, ..., which means b_m are real.

Let us represent $P_j^{(t)}(z_1)$ in a more convenient form.

For this, consider an auxiliary system of functions

$$\varphi_j^{(t)}(\lambda) = -\lambda^{s-1} \cdot \sum_{\nu=1}^{\infty}\frac{1}{w_{j(\nu)}^t} \cdot \frac{1}{1 - \frac{\lambda}{w_{1(\nu)}}} \cdot \prod_{\eta=1}^{\infty}\left(1 - \frac{\lambda}{w_{1(\eta)}}\right), \quad s \geqslant 1.$$

Or after the reduction:

$$\varphi_j^{(t)}(\lambda) = -\lambda^{s-1} \cdot \sum_{\nu=1}^{\infty}\frac{1}{w_{j(\nu)}^t} \cdot \prod_{\eta\neq\nu}\left(1 - \frac{\lambda}{w_{1(\eta)}}\right) = -\lambda^{s-1} \cdot \sum_{m=0}^{\infty} a_{jm}^{(t)} \cdot \lambda^m, \qquad a_{j0}^{(t)} = 1.$$

Using the geometric progression formula for sufficiently small $|\lambda|$:

$$\varphi_j^{(t)}(\lambda) = -\lambda^{s-1} \cdot \sum_{\nu=1}^{\infty}\frac{1}{w_{j(\nu)}^t} \cdot \sum_{m=0}^{\infty}\left(\frac{\lambda}{w_{1(\nu)}}\right)^m \cdot \prod_{\eta=1}^{\infty}\left(1 - \frac{\lambda}{w_{1(\eta)}}\right) =$$

$$= -\lambda^{s-1} \cdot \sum_{m=0}^{\infty}\lambda^m \cdot \left(\sum_{\nu=1}^{\infty}\frac{1}{w_{1(\nu)}^m \cdot w_{j(\nu)}^t}\right) \cdot \prod_{\eta=1}^{\infty}\left(1 - \frac{\lambda}{w_{1(\eta)}}\right) =$$

$$= -\lambda^{s-1} \cdot \left(\sum_{m=0}^{\infty} S_{me_1+te_j} \cdot \lambda^m \right) \cdot \left(\sum_{k=0}^{\infty} b_k \cdot \lambda^k \right) =$$

$$= -\lambda^{s-1} \cdot \sum_{l=0}^{\infty} \lambda^s \cdot \left(\sum_{m+k=l} S_{me_1+te_j} \cdot b_k \right),$$

where $S_{me_1+te_j} = \sum\limits_{\nu=1}^{\infty} \dfrac{1}{w^m_{1(\nu)} \cdot w^t_{j(\nu)}}$ are power sums for the multi-index $me_1 + te_j = (m, 0, \ldots, 0, t, 0, \ldots, 0)$, the first component of the multi-index is equal to 1, the j-th component is equal to t, and the remaining components are zeros.

Since the coefficients of the system (1) are real, we have that

$$\sum_{m+k=l}^{\infty} S_{m+k=l} \cdot b_k, \quad l = 0, 1, 2, \ldots$$

are real.

We obtained relations for calculating $a^{(t)}_{jl}$:

$$a^{(t)}_{jl} = \sum_{m+k=l} S_{me_1+te_j} \cdot b_k,$$

where $l = 0, 1, 2, \ldots$, $b_0 = 1$, $s \geqslant 1$, $t \geqslant 0$, $m \geqslant 0$, $k \geqslant 0$. That is, the coefficients $a^{(t)}_{jl}$ are real.

Thus, if one coordinate of the root of the system (1) is real, then all other coordinates of this root are also real. This is where the statement of the theorem follows. □

Corollary 2. *If a system (1) with real coefficients is such that all zeros of $Q(z_1)$ (that is, $R(z_1)$) are simple except for the point $z_1 = 0$ and the function $g(z_1)$ from (4) has real coefficients, then the number of real roots of the system (1) in $\mathcal{E}$ coincides with the number of real roots of the function $Q(z_1)$.*

The proof of Corollary 2 repeats the proof of Theorem 2.

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О вещественных корнях систем трансцендентных уравнений

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Аннотация. Работа посвящена нахождению числа вещественных корней систем трансцендентных уравнений. Показано, что если система имеет простые корни, то число вещественных координат корней одинаково. Поэтому число вещественных корней связано с числом вещественных корней результанта системы.

Ключевые слова: система трансцендентных уравнений, результант, вычетный интеграл.