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Some Properties of the Automorphisms of the Classical Domain of the First Type in the Space $\mathbb{C}[m \times n]$

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Abstract. In this article we obtain an analogue of Theorem 2.2.2 from Rudin's book [6] for classical Cartan domains of the first type.

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It is well known that thanks to Riemann's uniformization theorem an arbitrary simply connected domain whose boundary with more than one point is biholomorphically equivalent to a unit circle $U = \{z \in \mathbb{C} : |z| < 1\}$. But in $\mathbb{C}^n (n > 1)$ such a property does not hold. For example, a ball and a polidisk are not mutually biholomorphic equivalent. Therefore, the class of biholomorphic domains in the space $\mathbb{C}^n$ is very important.

Definition 1 (Homogeneous Domain)**.** *A domain $D \subset \mathbb{C}^n$ is called homogeneous if the group $Aut\,(D)$ of automorphisms of this domain is transitive, i.e. for any pair of points $z_1, z_2 \in D$ there exists an automorphism $\varphi \in Aut\,(D)$ such that $\varphi(z_1) = z_2$.*

Definition 2 (Symmetric Domain)**.** *A homogeneous domain $D \subset \mathbb{C}^n$ is called symmetric if for any point $\varsigma \in D$ there exists an automorphism $\varphi \in Aut\,(D)$such that:*
$\varphi(\varsigma) = \varsigma$ but $\varphi(z) \neq z$ for $z \neq \varsigma$;
$\varphi \circ \varphi = e$, where $e \in Aut\,(D)$ is the identity map.

Definition 3. *A domain $D \subset \mathbb{C}^n$ is called irreducible if it is not a direct product of bounded symmetric domains of lower dimension.*

Definition 4. *A bounded domain $D \subset \mathbb{C}^n$ is called classical if the complete group of its holomorphic automorphisms is a classical Lie group and it is transitive on it.*

In homogeneous domains, the automorphism groups ([1, 2]) can be used to find integral formulas. Domains with rich automorphism groups are often realized as matrix domains ([3, 4]). They turned out to be useful in solving various problems of function theory.

Complex homogeneous bounded domains are of great interest from different points of view. This is due to the fact that they form a relatively wide class of domains in $\mathbb{C}^n$, for which a number of meaningful, essentially multidimensional results have been obtained ([3, 5, 6] and etc.).

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In the works of C. Siegel, the presence of a biholomorphic mapping of classical domains by Siegel domains is shown [7]. Such biholomorphic maps are described, and applications to the problems of holomorphic maps to unbounded domains are given [8, 9]. Therefore, classical domains play an important role in multidimensional complex analysis. The goal of this work [10] is to obtain a criterion for holomorphic extendibility into a matrix ball for functions defined on a part of the Shilov boundary (skeleton) of a matrix ball, which is close in spirit to the criterion of L. A. Aizenberg, A. M. Kytmanov [11], and G. Khudayberganov [12]. In [13], optimal estimates of Bergman kernels for classical domains $\Re_I(m,n)$, $\Re_{II}(m)$, $\Re_{III}(m)$ and $\Re_{IV}(n)$ were found, respectively, through Bergman kernels in balls from spaces $\mathbb{C}^{mn}$, $\mathbb{C}^{m(m+1)/2}$, $\mathbb{C}^{m(m-1)/2}$ and $\mathbb{C}^n$. For this purpose, the statements of the Sommer-Mehring theorem on the continuation of the Bergman kernel and some properties of the Bergman kernel are used.

The theory of functions of many complex variables, or multidimensional complex analysis, currently has a fairly rigorously constructed theory [6, 14, 15]. At the same time, many questions of classical (one-dimensional) complex analysis still do not have unambiguous multidimensional analogues. This is due to the complex structure of a multidimensional complex space, overdetermination of the Cauchy-Riemann equations, the absence of a universal integral Cauchy formula, etc. In the works of E. Cartan, C. Siegel [7], Hua Lo-Ken [3], I. I. Pyatetsky-Shapiro [16] the matrix approach is widely used for presentation of the theory of multidimensional complex analysis.

E. Cartan (see [17]) in 1935 initiated a systematic study of bounded homogeneous domains, found all homogeneous domains in the spaces $\mathbb{C}^2$ and $\mathbb{C}^3$. He gave a classification of all bounded symmetric domains. These domains are divided into equivalence classes with respect to biholomorphic mappings. Each such class can be specified by any domain that belongs to it. Moreover, it is obvious that it is sufficient to consider only irreducible classes, that is, classes of domains that are not products of bounded symmetric domains of lower dimensions. In general, as E. Cartan established [17], there are six types of classes of irreducible bounded symmetric domains. Domains belonging to four of them are called classical because their automorphism groups are classical semisimple Lie groups. Two of these types are special in the sense that each of them occurs in the space $\mathbb{C}^n$ of only one dimension n, respectively for $n = 16$ and $n = 27$.

Consider the classical domains (see. [3, 17]):

$$\Re_I(m,n) = \left\{ Z \in \mathbb{C}[m,n] : I^{(m)} - Z\bar{Z}' > 0 \right\},$$

$$\Re_{II}(m) = \left\{ Z \in \mathbb{C}[m,m] : I^{(m)} - Z\bar{Z} > 0, \forall Z' = Z \right\},$$

$$\Re_{III}(m) = \left\{ Z \in \mathbb{C}[m,m] : I^{(m)} + Z\bar{Z} > 0, \forall Z' = -Z \right\},$$

$$\Re_{IV}(n) = \left\{ Z \in \mathbb{C}^n : |\langle z,z\rangle|^2 - 2|z|^2 + 1 > 0, \, |\langle z,z\rangle| < 1 \right\},$$

here $I^{(m)}$ is the identity matrix of order m, $\bar{Z}'$ is the complex conjugate matrix of the transposed matrix Z' ($H > 0$ for a Hermitian matrix H means, as usual, that H is positive definite). All these domains are homogeneous symmetric convex complete circular domains centered at O (O is the zero matrix).

If we write the elements of the matrices $Z \in \mathbb{C}[m,n]$ as a point in the space $\mathbb{C}^{mn}$ then it is in the following form

$$z = \{z_{11}, \ldots, z_{1n}, z_{21}, \ldots, z_{2n}, \ldots, z_{m1}, \ldots, z_{mn}\} \in \mathbb{C}^{mn}.$$

Then we can assume that Z is an element of the space in $\mathbb{C}^{mk}$, i.e., we arrive to the following isomorphism

$$\mathbb{C}[m \times n] \cong \mathbb{C}^{mn}.$$

Therefore, the dimensions of the classical four domains above are equal to

$$mn,\ \frac{m(m+1)}{2},\ \frac{m(m-1)}{2},\ n$$

respectively.

In [18] an analogue of Bremermann's theorem on finding the Bergman kernel is obtained for the Cartesian product of classical domains.

Writing out explicity the transitive group of automorphisms of four types of classical domains and matrix balls (see, for example, [19, 20]) associated with classical domains. By direct computation one can find the Bergman and Cauchy-Szegő kernels for these domains. And then (using the properties of the Poisson kernel), we can find the Carleman formula, which restores the value of a holomorphic function in the domain itself from its values on some boundary sets of uniqueness (see [21–24]). In this case, the scheme for finding the Bergman and Cauchy–Szegő kernels from [3, 6, 25] is used.

The properties of the matrix ball $B_2(m,n)$ of the second type are studied in [26]. In [27] the volumes of a matrix ball of the third type and a generalized Lie ball are calculated. The full volumes of these domains are necessary to find the kernels of integral formulas for these domains (Bergman, Cauchy–Szegő, Poisson kernels, etc.) and is used for the integral representation of functions holomorphic in these domains, in the mean value theorem, and in other important concepts.

For example, in [28] the regularity and algebraicity of mappings in classical domains are studied, and in [29] harmonic Bergman functions in classical domains are studied from a new point of view.

The first type of classical domain: $\Re_1(m,n)$. Each point of the domain is a matrix with m rows and n columns satisfying the following condition

$$I^{(m)} - ZZ^* > 0,$$

where $I^{(m)}$ is a unit square matrix of the m^{th} order and Z^* is a transposed conjugate matrix for Z. Automorphisms of the classical domain of the first type $\Re_1(m,n)$ have the following form [3]

$$\varphi(Z) = (AZ+B)(CZ+D)^{-1}, \tag{1}$$

where the coefficients satisfy the following conditions:

$$AA^* - BB^* = I^{(m)},\ \ AC^* = BD^*,\ \ CC^* - DD^* = -I^{(n)}. \tag{2}$$

The automorphism (1) of the classical domain can also be represented as

$$\varphi(Z) = (ZB^* + A^*)^{-1}(ZD^* + C^*), \tag{3}$$

where the coefficients (3) satisfy the following conditions:

$$B^*B - D^*D = -I^{(n)},\ \ B^*A = D^*C,\ \ A^*A - C^*C = I^{(m)}. \tag{4}$$

We can simplify the automorphism (1) as follows

$$\varphi(Z) = (AZ+B)(CZ+D)^{-1} = A\left(Z + A^{-1}B\right)\left(D^{-1}CZ + I\right)^{-1}D^{-1}. \tag{5}$$

If we put $A = Q,\ A^{-1}B = -P,\ D = R$, then we have

$$\varphi(Z) = \varphi_P(Z) = Q(Z-P)(I - P^*Z)^{-1}R^{-1}. \tag{6}$$

Then from (2) we get

$$Q(I - PP^*)Q^* = I^{(m)},\ \ R(I - P^*P)R^* = I^{(n)}. \tag{7}$$

We study some useful properties of automorphism $\varphi_P(Z)$ in the following theorem (this is an analog of the theorem 2.2.2 from [6]).

Theorem 1. *For an automorphism $\varphi_P(Z) = Q(Z-P)(I-P^*Z)^{-1}R^{-1}$ of the classical domain of the first type $\Re_1(m,n)$, the following properties hold:*

1^0. *$\varphi_P(P) = 0$, $\varphi_P(0) = -QPR^{-1}$. In particular, if $QP + PR = 0$ then we have $\varphi_P(P) = 0$, $\varphi_P(0) = P$.*

2^0. *Differential of the automorphism of the domain $\Re_1(m,n)$ is equal to*

$$d(\varphi_P(P)) = QdZR^*, d(\varphi_P(0)) = (Q^*)^{-1}dZR^{-1},$$

where $d(\varphi_P(P))$ $(d(\varphi_P(0)))$ is the differential of the automorphism (6) at the point $Z = P$ $(Z = 0)$.

3^0. *For all $Z, W \in \Re_I(m,n)$ we have*

$$\det(I - \langle\varphi(Z), \varphi(W)\rangle) = \frac{\det(I - \langle P, P\rangle) \cdot \det((I - \langle Z, W\rangle))}{\det(I - \langle Z, P\rangle) \cdot \det(I - \langle P, W\rangle)},$$

$$\det(I - \langle\varphi_P(Z), \varphi_P(Z)\rangle) = \frac{\det(I - \langle P, P\rangle)\det(I - \langle Z, Z\rangle)}{\det(I - \langle Z, P\rangle)\det(I - \langle P, Z\rangle)}.$$

4^0. *If we have the following equalities $QP + PR = 0$, $R = R^*$, $Q = Q^*$ then $\varphi_P(\varphi_P(Z)) = Z$ (the property of involution).*

5^0. *Finally, $\varphi_P(Z)$ is a homeomorphism.*

Proof. Let us prove turn by turn the above properties of the automorphism (6).

1^0. The values of the automorphism $\varphi_P(Z)$ at the points $Z = P$ and $Z = 0$ are

$$\varphi_P(P) = Q(P-P)(I-P^*P)^{-1}R^{-1} = 0,$$

$$\varphi_P(0) = Q(0-P)(I-P^*\cdot 0)^{-1}R^{-1} = -QPR^{-1},$$

that is

$$\varphi_P(P) = 0,\ \varphi_P(0) = -QPR^{-1}$$

the latter equalities follow directly. The condition $QP + PR = 0$ implies $QP = -PR \Rightarrow P = -QPR^{-1}$, that is,

$$\varphi_P(0) = P.$$

We show that the matrix $-QPR^{-1}$ belongs to $\Re_1(m,n)$:

$$I - \left(-QPR^{-1}\right)\cdot\left(-QPR^{-1}\right)^* = I - QPR^{-1}\left(R^{-1}\right)^*P^*Q^*.$$

Thanks to the conditions (7) we have

$$R^{-1}(R^*)^{-1} = I - P^*P.$$

$I - \left(-QPR^{-1}\right)\cdot\left(-QPR^{-1}\right)^* = I - QPR^{-1}\left(R^{-1}\right)^*P^*Q^* = I - QP(I-P^*P)P^*Q^* =$
$= I - QPP^*Q^* + QPP^*PP^*Q^* = I - QPP^*(I-PP^*)Q^* = I - QPP^*Q^{-1}Q(I-PP^*)Q^* =$
$= I - QPP^*Q^{-1}I^{(m)} = (Q - QPP^*)Q^{-1} = Q(I-PP^*)Q^{-1} = Q(I-PP^*)Q^*(Q^*)^{-1}Q^{-1} =$
$= I^{(m)}(Q^*)^{-1}Q^{-1} = (Q^*)^{-1}Q^{-1} = (Q^*)^{-1}I^{(m)}Q^{-1} > 0$

Therefore, $\left(-QPR^{-1}\right) \in \Re_1(m,n)$ [30].

2^0. Now we calculate the differential of the automorphism $\varphi_P(Z) = Q(Z-P)(I-P^*Z)^{-1}R^{-1}$. So we have

$$d(\varphi_P(Z)) = Q\left(dZ(I-P^*Z)^{-1} + (Z-P)\,d(I-P^*Z)^{-1}\right)R^{-1} \tag{8}$$

or

$$\varphi_P(Z)R(I-P^*Z) = Q(Z-P).$$

From the last equality and according to the rules of differentiation we get

$$d\left(\varphi_P\left(Z\right)\right)R\left(I-P^*Z\right)+\varphi_P\left(Z\right)Rd\left(I-P^*Z\right)=Qd\left(Z-P\right),$$
$$d\left(\varphi_P\left(Z\right)\right)R\left(I-P^*Z\right)-\varphi_P\left(Z\right)RP^*dZ=QdZ,$$
$$d\left(\varphi_P\left(Z\right)\right)R\left(I-P^*Z\right)=QdZ+\varphi_P\left(Z\right)RP^*dZ,$$
$$d\left(\varphi_P\left(Z\right)\right)R\left(I-P^*Z\right)=\left(Q+\varphi_P\left(Z\right)RP^*\right)dZ,$$
$$d\left(\varphi_P\left(Z\right)\right)R\left(I-P^*Z\right)=\Big(Q+Q\left(Z-P\right)\left(I-P^*Z\right)^{-1}R^{-1}RP^*\Big)\,dZ.$$

Then we obtain

$$d\left(\varphi_P\left(Z\right)\right)=Q\Big(I+\left(Z-P\right)\left(I-P^*Z\right)^{-1}P^*\Big)\,dZ(I-P^*Z)^{-1}R^{-1}. \tag{9}$$

If we calculate the differential of $\varphi_P\left(Z\right)$ at the points $Z=P$ and $Z=0$, we get the following values

$$d\left(\varphi_P\left(P\right)\right)=QdZ(I-P^*P)^{-1}R^{-1},\quad d\left(\varphi_P\left(0\right)\right)=Q(I-PP^*)dZR^{-1}.$$

In accordance with the conditions (7), the following equality

$$(I-P^*P)^{-1}R^{-1}=R^*,\quad Q(I-PP^*)=(Q^*)^{-1}.$$

Then for the differentials of $\varphi_P\left(Z\right)$ at the points $Z=P$ and $Z=0$ we have

$$d\left(\varphi_P\left(P\right)\right)=QdZR^*,\quad d\left(\varphi_P\left(0\right)\right)=(Q^*)^{-1}dZR^{-1}$$

3^0. In order to prove this property, we use the expression (3) of the automorphism of the domain $\Re_1\left(m,n\right)$. $\varphi_P\left(Z\right)=\left(ZB^*+A^*\right)^{-1}\left(ZD^*+C^*\right)$, $\varphi_P\left(W\right)=\left(WB^*+A^*\right)^{-1}\left(WD^*+C^*\right)$

$$\begin{aligned}
I-\left\langle\varphi\left(Z\right),\varphi\left(W\right)\right\rangle &= I-\left(ZB^*+A^*\right)^{-1}\left(ZD^*+C^*\right)\Big(\left(WB^*+A^*\right)^{-1}\left(WD^*+C^*\right)\Big)^*=\\
&=I-\left(ZB^*+A^*\right)^{-1}\left(ZD^*+C^*\right)\left(DW^*+C\right)\left(BW^*+A\right)^{-1}=\\
&=\left(ZB^*+A^*\right)^{-1}\big(\left(ZB^*+A^*\right)\left(BW^*+A\right)-\left(ZD^*+C^*\right)\left(DW^*+C\right)\big)\left(BW^*+A\right)^{-1}=\\
&=\left(ZB^*+A^*\right)^{-1}\big(ZB^*BW^*+ZB^*A+A^*BW^*+A^*A-(ZD^*DW^*+ZD^*C+\\
&\qquad\qquad +C^*DW^*+C^*C)\big)(BW^*+A)^{-1}=\\
&=\left(ZB^*+A^*\right)^{-1}\left(I-ZW^*\right)\left(BW^*+A\right)^{-1}=\\
&=(A^*)^{-1}\Big(ZB^*(A^*)^{-1}+I\Big)^{-1}\left(I-ZW^*\right)\left(A^{-1}BW^*+I\right)^{-1}A^{-1}.
\end{aligned}$$

If we put

$$A=Q,\ A^{-1}B=-P,\ D=R$$

and use

$$A^*=Q^*,\ B^*\left(A^{-1}\right)^*=-P^*,\ D^*=R^*,$$

we get the following equality

$$\begin{aligned}
I-\left\langle\varphi\left(Z\right),\varphi\left(W\right)\right\rangle &= (Q^*)^{-1}(I-ZP^*)^{-1}\left(I-ZW^*\right)\left(I-PW^*\right)^{-1}Q^{-1}=\\
&=(Q^*)^{-1}(I-\langle Z,P\rangle)^{-1}\left(I-\langle Z,W\rangle\right)\left(I-\langle P,W\rangle\right)^{-1}Q^{-1}.
\end{aligned} \tag{10}$$

By using (7), it is not difficult to see that

$$\det\left(Q\left(I-PP^*\right)Q^*\right)=\det\left(I^{(m)}\right),\quad \det\left(\left(I-PP^*\right)\right)=\det\left(Q^{-1}\right)\cdot\det\left(\left(Q^*\right)^{-1}\right).$$

We use the last two equalities to obtain the following relation

$$\det\left(I-\langle\varphi\left(Z\right),\ \ \varphi\left(W\right)\rangle\right)=\frac{\det\left(I-\langle P,P\rangle\right)\cdot\det\left(\left(I-\langle Z,W\rangle\right)\right)}{\det\left(I-\langle Z,P\rangle\right)\cdot\det\left(I-\langle P,W\rangle\right)} \tag{11}$$

If we put $W=Z$ in (11), we get

$$\det\left(I-\langle\varphi_P\left(Z\right),\ \ \varphi_P\left(Z\right)\rangle\right)=\frac{\det\left(I-\langle P,P\rangle\right)\det\left(I-\langle Z,Z\rangle\right)}{\det\left(I-\langle Z,P\rangle\right)\det\left(I-\langle P,Z\rangle\right)}.$$

$\mathbf{4}^0$. We show that the automorphism $\varphi_P\left(Z\right)=Q\left(Z-P\right)\left(I-P^*Z\right)^{-1}R^{-1}$ is an involution of the domain $\Re_1\left(m,n\right)$. Indeed,

$$\begin{aligned}\varphi_P\left(\varphi_P\left(Z\right)\right)&=Q\left(Q\left(Z-P\right)\left(I-P^*Z\right)^{-1}R^{-1}-P\right)\left(I-P^*Q\left(Z-P\right)\left(I-P^*Z\right)^{-1}R^{-1}\right)^{-1}R^{-1}=\\&=Q\left(Q\left(Z-P\right)-PR\left(I-P^*Z\right)\right)\left(I-P^*Z\right)^{-1}R^{-1}\cdot R\cdot\left(I-P^*Z\right)\times\\&\qquad\qquad\times\left(R\left(I-P^*Z\right)-P^*Q\left(Z-P\right)\right)^{-1}R^{-1}=\\&=Q\left(QZ-QP-PR+PRP^*Z\right)\cdot\left(R-RP^*Z-P^*QZ+P^*QP\right)^{-1}R^{-1}=\\&=Q\left(\left(Q+PRP^*\right)Z-\left(QP+PR\right)\right)\cdot\left(\left(R+P^*QP\right)-\left(RP^*+P^*Q\right)Z\right)^{-1}R^{-1}.\end{aligned}$$

Since $QP+PR=0$, $R=R^*$, $Q=Q^*$ and considering the following equalities

$$Q\left(I-PP^*\right)Q^*=I^{(m)},\quad R\left(I-P^*P\right)R^*=I^{(n)},$$

we have

$$Q\left(Q+PRP^*\right)=I^{(m)},\quad RP^*+P^*Q=0,\quad \left(R+P^*QP\right)^{-1}R^{-1}=I^{(n)}.$$

Consequently, we obtain

$$\varphi_P\left(\varphi_P\left(Z\right)\right)=Q\left(\left(Q+PRP^*\right)Z-\left(QP+PR\right)\right)\cdot\left(\left(R+P^*QP\right)-\left(RP^*+P^*Q\right)Z\right)^{-1}R^{-1}=Z,$$

i.e

$$\varphi_P\left(\varphi_P\left(Z\right)\right)=Z.$$

$\mathbf{5}^0$. In order to prove this property we take $Z_1,Z_2\in\Re_1$. Then

$$\varphi_P\left(Z_1\right)=Q\left(Z_1-P\right)\left(I-P^*Z_1\right)^{-1}R^{-1},\varphi_P\left(Z_2\right)=Q\left(Z_2-P\right)\left(I-P^*Z_2\right)^{-1}R^{-1},$$

$$\begin{aligned}\varphi_P\left(Z_1\right)-\varphi_P\left(Z_2\right)&=Q\left(Z_1-P\right)\left(I-P^*Z_1\right)^{-1}R^{-1}-Q\left(Z_2-P\right)\left(I-P^*Z_2\right)^{-1}R^{-1}=\\&=Q\left(\left(Z_1-P\right)\left(I-P^*Z_1\right)^{-1}-\left(Z_2-P\right)\left(I-P^*Z_2\right)^{-1}\right)R^{-1}=\end{aligned}$$

$$=Q\left(Z_1(I-P^*Z_1)^{-1}-Z_2(I-P^*Z_2)^{-1}-P\left((I-P^*Z_1)^{-1}-(I-P^*Z_2)^{-1}\right)\right)R^{-1}=$$

$$\begin{aligned}=Q\Big(&Z_1(I-P^*Z_1)^{-1}-Z_2(I-P^*Z_1)^{-1}+Z_2(I-P^*Z_1)^{-1}-\\&-Z_2(I-P^*Z_2)^{-1}-P\left((I-P^*Z_1)^{-1}-(I-P^*Z_2)^{-1}\right)\Big)R^{-1}=\end{aligned}$$

$$= Q\left(\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1} + \left(Z_2 - P\right)\left(\left(I - P^*Z_1\right)^{-1} - \left(I - P^*Z_2\right)^{-1}\right)\right)R^{-1} =$$

$$= Q\left(\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1} + (Z_2 - P)(I - P^*Z_2)^{-1}(I - P^*Z_2 - I + P^*Z_1)(I - P^*Z_1)^{-1}\right)R^{-1} =$$

$$= Q\left(\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1} + \left(Z_2 - P\right)\left(I - P^*Z_2\right)^{-1}P^*\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1}\right)R^{-1} =$$

$$= Q\left(I + \left(Z_2 - P\right)\left(\left(P^*\right)^{-1} - \left(P^*\right)^{-1}P^*Z_2\right)^{-1}\right)\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1}R^{-1} =$$

$$= Q\left(\left(P^*\right)^{-1} - Z_2 + \left(Z_2 - P\right)\right)\left(\left(P^*\right)^{-1} - Z_2\right)^{-1}\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1}R^{-1} =$$

$$= Q\left(\left(P^*\right)^{-1} - Z_2 + \left(Z_2 - P\right)\right)\left(\left(P^*\right)^{-1} - Z_2\right)^{-1}\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1}R^{-1} =$$

$$= Q\left(I - PP^*\right)\left(P^*\right)^{-1}P^*\left(I - Z_2P^*\right)^{-1}\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1}R^{-1} =$$

$$= Q\left(I - PP^*\right)\left(I - Z_2P^*\right)^{-1}\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1}R^{-1}.$$

Thus

$$\varphi_P\left(Z_1\right) - \varphi_P\left(Z_2\right) = Q\left(I - PP^*\right)\left(I - Z_2P^*\right)^{-1}\left(Z_1 - Z_2\right)\left(I - P^*Z_1\right)^{-1}R^{-1}.$$

Since $Q\left(I - PP^*\right)\left(I - Z_2P^*\right)^{-1}$ and $\left(I - P^*Z_1\right)^{-1}R^{-1}$ are different from O(O is the zero matrix), we can easily see that $\varphi_P\left(Z_1\right) = \varphi_P\left(Z_2\right)$ if and only if $Z_1 = Z_2$. Hence $\varphi_P\left(Z\right)$ is a homeomorphism.

The proof is complete. □

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Некоторые свойства автоморфизмов классической области первого типа в пространстве $\mathbb{C}[m\times n]$

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Аннотация. В этой статье получен аналог Теоремы 2.2.2 из книги Рудина [6] для классических областей Картана первого типа.

Ключевые слова: однородная область, симметричная область, классическая область, автоморфизм.